Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 196 (2007) 2541–2554
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

# Convergence of a MFE–FV method for two phase flow with applications to heap leaching of copper ores

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Received 26 January 2006; received in revised form 9 August 2006; accepted 8 November 2006

**Abstract**

In this paper we describe error estimates for a finite element approximation to partial differential systems describing two-phase immiscible flows in porous media, with applications to heap leaching of copper ores. These approximations are based on mixed finite element (MFE) methods for the pressure and velocity and finite volume (FV) for the saturation. The fluids are considered incompressible. Numerical results for heap leaching simulation are presented.
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*Keywords:* Convection dominated-diffusion problem; Two phase flow in porous media; Combined MFE–FV; A priori error estimates; Heap leaching

## 1. Introduction

We can use the knowledge, experience and physical intuition accumulated in the hydrological sciences and petroleum engineering to simulate, optimize and improve heap leaching operations today. Leaching is a mass transfer process between the leaching solution (fluid phase) and the ore bed (solid phase) [1,2]. The heap leaching process can be considered as a multiphase flow phenomenon in a porous medium, where the fluid phase is composed by a liquid (leach solution) and a gas [3,4]. Two distinct phenomena are of interest in the study of heap leaching: the fluid flow and the physicochemical reactions [5]. These two phenomena can be studied separately if the extent of leaching does not influence the flow pattern. In other words, the flow pattern in a heap depends on the initial conditions of the heap only. In general, researchers in heap leaching have separated the fluid flow problem from the physicochemical problem.

In this paper, we study the convergence of a numerical scheme for the fluid flow model. We use the classical two-phase flow equations, which can be rewritten in differential formulations so that the coupling and nonlinearity are weakened. These formulations include, phase, global, and weighted formulations. We consider the global formulation, specifically, the fractional flow formulation for two-phase immiscible and incompressible fluids.

It is well known that advective transport in diffusive effects dominates for two-phase flow equations in porous media. Hence, it is important to obtain accurate approximate fluid velocities. This motivates the use of mixed finite element methods for the computation of pressure and velocity, due to the convection–diffusion control of the saturation equation, efficient and accurate approximations should be used to solve this equation. On the other hand, finite volume methods should be considered for the computation of the leaching equation, resolving shock fronts in a proper manner.

MFE–FV schemes for two phase flow models were first proposed by Durlofsky [6] (see also [7]) without a

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doi:10.1016/j.cma.2006.11.022

convergence analysis. A results of convergence for a particular case of two phase flow system, with linear flux, non-degeneracity of the diffusion terms, and without gravitational effects, were proved by Ohlberger [8]. A fully discrete finite element analysis of multiphase flow in groundwater hydrology was given by Chen and Ewing [9] for smooth solutions of for fractional flow formulation, with a constant liquid density and a gaseous density depending on the global pressure. An error estimates for finite approximations of the system, which are based on MFE methods for pressure and velocity and characteristic finite element methods for saturation was proved by Chen [10]. A procedure which consisted in a MFE method for pressure equation and an upwind scheme was considered by Chavent and Jaffré [11]. It is based on a discontinuous finite element approximation associated with a slope limiter for the saturation equation. In degenerate cases, *i.e.*, when the diffusion term becomes zero for some saturation values, Chen and Ewing [12] considered a finite element approximation where the elliptic equation for the pressure and velocity is approximated by a mixed finite element method, while the degenerate parabolic equation for the saturation is approximated by a Galerkin finite element method.

A more detailed and extensive review of different numerical methods for classical two phase equations, for immiscible and incompressible flow, can be found in the paper of Chavent and Jaffré [11], in the reservoir simulation context, and in the paper of Helmig [13], in the environmental engineering context.

The aim of this paper is to study convergence for the two phase flow system with applications to heap leaching of copper ores. This is done by proving an a priori error estimate. Our proof follows the main ideas of Ohlberger [8]. But, additionally, our model consider a nonlinear convective term and a nonlinear gravitational term both of which are very important in heap leaching, because the flow is mainly vertical. In contrast to [8], our problem consider non-homogeneous Neumann boundary conditions, which corresponds to the physical behavior of the irrigation and infiltration processes in Heap Leaching. Finally, we obtain numerical results, with experimental parameters from the copper industry in Chile.

The paper is organized as follows. In Section 2, we state the continuous problem. In Section 3 we state the discrete problem. In Section 4 we present the main convergence results. In Section 5 we develop some preliminary results, which will be useful in the convergence analysis. In Section 6 we proof the convergence of the semi discrete scheme. In Section 7 we proof the convergence of the fully discrete scheme. Finally, in Section 8 we present results of the numerical experiments.

## 2. Statement of the continuous problem

In this section we present the classical two phase immiscible and incompressible flow equations for the fluid flow problem in the context of Heap Leaching. Next, we define a fractional flow formulation for the degenerate and non-degenerate case in a weak form. Finally, we define a model problem for our convergence analysis.

### 2.1. Physical problem

In this paper we consider two dimensional geometry, *i.e.*, a transversal cut of the heap (Fig. 1). The boundary $\partial\Omega$ of the domain $\Omega \subset \mathbb{R}^2$ is expressed as $\partial\Omega = \Gamma^i \cup \Gamma^o \cup \Gamma^{\mathrm{l}} \cup \Gamma^{\mathrm{r}}$, where $\Gamma^i$ is the input boundary (zone of irrigation), $\Gamma^o$ is the output boundary (zone of drainage), $\Gamma^{\mathrm{l}}$ is the left boundary and $\Gamma^{\mathrm{r}}$ is the right boundary. In particular, in the context of heap leaching, we can assume that the porosity $\phi$, and the densities $\rho_{\mathrm{w}}$ and $\rho_{\mathrm{n}}$ are constants, that there are no source terms $q_{\mathrm{w}} = q_{\mathrm{n}} = 0$, and that $\mathbf{K} = kI$ represents the intrinsic permeability tensor as a characteristic property of the porous matrix only. Therefore, the physical problem for the fluid flow in heap leaching process is given by the following system (see [14] and [13]):

$$\phi \frac{\partial s_{\mathrm{w}}}{\partial t} + \nabla \cdot \mathbf{v}_{\mathrm{w}} = 0, \tag{1}$$

$$\phi \frac{\partial s_{\mathrm{n}}}{\partial t} + \nabla \cdot \mathbf{v}_{\mathrm{n}} = 0, \tag{2}$$

$$\mathbf{v}_{\mathrm{w}} = -k\frac{k_{\mathrm{rw}}}{\mu_{\mathrm{w}}}(\nabla p_{\mathrm{w}} - \rho_{\mathrm{w}}\mathbf{g}), \tag{3}$$

$$\mathbf{v}_{\mathrm{n}} = -k\frac{k_{rn}}{\mu_{\mathrm{n}}}(\nabla p_{\mathrm{n}} - \rho_{\mathrm{n}}\mathbf{g}), \tag{4}$$

$$p_{\mathrm{c}}(s_{\mathrm{w}}) = p_{\mathrm{n}} - p_{\mathrm{w}}, \tag{5}$$

$$s_{\mathrm{w}} + s_{\mathrm{n}} = 1 \tag{6}$$

for all $\mathbf{x} \in \Omega$, and $t > 0$, where $s_\alpha$ is the saturation, with $\alpha = \mathrm{w}$ denoting the leaching solution and $\alpha = \mathrm{n}$ denoting the gaseous phase, $\mathbf{v}_\alpha$ is the volumetric velocity, $\mu_\alpha$ is the viscosity, $k_{\mathrm{r}\alpha}$ is the relative permeability, $\mathbf{g} = (0, -g)$, $g = 9.8\ [\mathrm{m/s^2}]$, is the gravitational, downward-pointing, constant vector, and $p_{\mathrm{c}}$ is the capillary pressure. Additionally, we assume the following initial conditions:

$$s_{\mathrm{w}}(\mathbf{x}, 0) = s_{\mathrm{w}}^o, \quad p_{\mathrm{n}}(\mathbf{x}, 0) = p_A$$

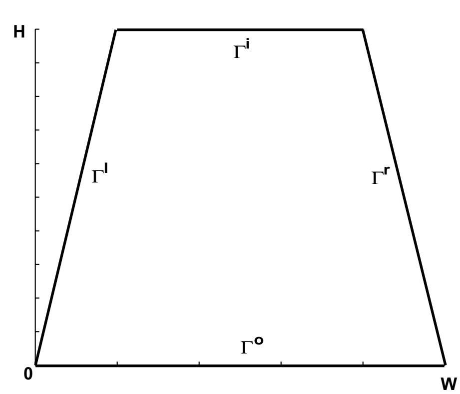


Fig. 1. Mathematical domain (transversal cut of the heap).

for all $\mathbf{x} \in \Omega$, where $p_\alpha$ is the pressure, $s_w^o$ is the initial saturation, $p_A$ is the atmospheric pressure, and we assume the boundary conditions

$$(\mathbf{v}_w \cdot \mathbf{n})(\mathbf{x}, t) = -R, \quad \mathbf{x} \in \Gamma^i,$$
$$(\mathbf{v}_w \cdot \mathbf{n})(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \Gamma^r \cup \Gamma^l,$$
$$(\nabla p_w \cdot \mathbf{n})(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \Gamma^o,$$
$$(\nabla s_n \cdot \mathbf{n})(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \Gamma^r \cup \Gamma^l \cup \Gamma^i,$$
$$(\mathbf{v}_n \cdot \mathbf{n})(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \Gamma^o$$

for all $t > 0$, and where $R = R(t) \geqslant 0$ is the irrigation ratio. In what follows we will omit $w$ in $s_w$.

### 2.2. Fractional flow formulation

Eqs. (1)–(6) can be rewritten in a different differential formulations so that the coupling and nonlinearity are weakened. This paper follows the fractional flow formulation [11], *i.e.*, a formulation in terms of a saturation and a global pressure. The main reason for this fractional flow approach is that efficient numerical methods can be devised to take advantage of the many physical properties inherent in to flow equations [9]. Now we introduce the total mobility $\lambda(s) := \lambda_w + \lambda_n$, where $\lambda_\alpha(s) := k_{r\alpha}/\mu_\alpha$ are the phase mobilities for $\alpha = w, n$, $f_\alpha(s) := \lambda_\alpha/\lambda$ the fractional flow functions, and the total velocity given by $\mathbf{u} := \mathbf{v}_w + \mathbf{v}_n$. Note that adding (1) and (2), and using (6), we obtain $\nabla \cdot \mathbf{u} = 0$. Additionally, following [11], we define the global pressure

$$p := p_n - \int_0^s (f_w p_c')(\mathbf{x}, \xi)\, d\xi, \tag{7}$$

noting that $\nabla p = \nabla p_n - f_w \nabla p_c$.

#### 2.2.1. Weakly degenerate formulation

Summing (3) and (4), and using the gradient computation of (7), we obtain the total velocity

$$\mathbf{u} = \mathbf{v}_w + \mathbf{v}_n = -k\lambda(\nabla p - G_\lambda \mathbf{g}), \tag{8}$$

where $G_\lambda := (\lambda_w \rho_w + \lambda_n \rho_n)/\lambda$. On the other hand, manipulating Eqs. (3) and (4), we obtain $\lambda_n \mathbf{v}_w - \lambda_w \mathbf{v}_n = \lambda_n \lambda_w (\nabla p_c + (\rho_w - \rho_n)\mathbf{g})$, and using (8), we deduce

$$\mathbf{v}_w = f_w(s)\mathbf{u} - D_w(s)\nabla s - G_w(s)\mathbf{g}, \tag{9}$$
$$\mathbf{v}_n = f_n(s)\mathbf{u} - D_n(s)\nabla s_n + G_n(s)\mathbf{g}, \tag{10}$$

where

$$D_w(s) := -k\lambda_n(s) f_w(s) p_c'(s),$$
$$D_n(s) := -k\lambda_w(s) f_n(s) p_c'(s),$$
$$G_w(s) := -k\lambda_n(s) f_w(s)(\rho_w - \rho_n),$$
$$G_n(s) := -k\lambda_w(s) f_n(s)(\rho_w - \rho_n).$$

Therefore, collecting (8)–(10) we define an alternative formulation for the system (1)–(6) which is called Fractional Flow Formulation

$$\nabla \cdot \mathbf{u} = 0, \tag{11}$$
$$\mathbf{u} = -k\lambda(\nabla p - G_\lambda \mathbf{g}), \tag{12}$$
$$\phi \frac{\partial s}{\partial t} = -\nabla \cdot (f_w \mathbf{u} - D_w \nabla s - G_w \mathbf{g}) \tag{13}$$

for all $\mathbf{x} \in \Omega$ and $t > 0$, with the initial conditions

$$s(\mathbf{x}, 0) = s_w^o \quad p(\mathbf{x}, 0) = p_o \tag{14}$$

for all $\mathbf{x} \in \Omega$, and the boundary conditions

$$(\mathbf{u} \cdot \mathbf{n})(\mathbf{x}, t) = \varphi_1(s(\mathbf{x}, t)), \quad (\mathbf{v}_w \cdot \mathbf{n})(\mathbf{x}, t) = \varphi_2(s(\mathbf{x}, t)) \tag{15}$$

for all $\mathbf{x} \in \Gamma$ and $t > 0$, where the functions $\varphi_1$ and $\varphi_2$ are know from previous expressions. Note that the Eq. (13) is parabolic and weakly degenerate, because $D_w(s_{wr}) = 0$ and $D_w(1) = 0$, where $s_{wr}$ is the residual saturation for the liquid phase.

#### 2.2.2. Non-degenerate formulation

Rather than a saturation, a complementary pressure was introduced by Chen [15]. In this form, the system formally appears to be non-degenerate. In effect, the complementary pressure, *i.e.*, the Kirchhoff transformation, is defined as

$$\theta := -\int_0^s (\lambda_n f_w p_c')(\mathbf{x}, \xi)\, d\xi, \tag{16}$$

where $s$ is related to $\theta$ through $s = \mathscr{S}(\theta)$, where $\mathscr{S}(\mathbf{x}, \theta)$ is the inverse of (16) for $0 \leqslant \theta \leqslant \theta^*$ with $\theta^*(\mathbf{x}) := -\int_0^1 \lambda_n f_w p_c'(\mathbf{x}, \xi)\, d\xi$. From this definition we obtain alternatives expressions for $\mathbf{u}$, $\mathbf{v}_w$ and $\mathbf{v}_n$, given by

$$\mathbf{u} = -k(\lambda(s)\nabla p + \gamma_1'(s)),$$
$$\mathbf{v}_w = -k(\nabla\theta + \lambda_w(s)\nabla p + \gamma_2'(s)) = f_w(s)\mathbf{u} - k\nabla\theta - k\gamma_2(s),$$
$$\mathbf{v}_n = k(\nabla\theta - \lambda_n(s)\nabla p + \gamma_3'(s)),$$

where the definition of $\gamma_i'$, $i = 1, 2, 3$ and $\gamma_2$ can be found in [15] and [12]. Therefore, we obtain an non-degenerate alternative formulation for the system Eqs. (1)–(6) given by

$$\nabla \cdot \mathbf{u} = 0, \tag{17}$$
$$\mathbf{u} = -k(\lambda\nabla p + \gamma_1'), \tag{18}$$
$$\phi \frac{\partial s}{\partial t} = -\nabla \cdot (f_w(s)\mathbf{u} - k\nabla\theta - k\gamma_2(s)), \tag{19}$$

in the unknowns $\mathbf{u}$, $p$, and $\theta$, with the initial and boundary conditions similar to (14) and (15). The differential system has a clear structure; the pressure equation is elliptic for $p$ and the saturation equation is parabolic for $\theta$ (degenerate for $s$). Its mathematical properties such as existence, uniqueness, regularity and asymptotic behavior of solution have been studied by Chen [15,16].

### 2.3. Weak formulation

Define the spaces as

$$\mathbf{V}(g) := \{\mathbf{v} \in H(\mathrm{div}; \Omega) | \mathbf{v} \cdot \mathbf{n} = g, \partial\Omega\},$$
$$W := \left\{ v \in L^2(\Omega) \middle| \int_\Omega v(\mathbf{x}, t)\, d\mathbf{x} = 0 \right\},$$

and $M := H^1(\Omega)$. Define the bilinear forms $A$ and $B$ as $A(\xi;\mathbf{v},\mathbf{w}) = \int_\Omega a(\xi)\mathbf{v}\cdot\mathbf{w}$ and $B(\mathbf{v},\varphi) := -\int_\Omega \varphi\nabla\cdot\mathbf{v}$. Introducing the weak form of the system (17)–(19): find $\mathbf{u}\in L^\infty(J;V(\varphi_1))$, $p\in L^\infty(J;W)$, and $\theta\in L^2(J;M)$ such that $s=\mathscr{S}(\theta)$, $\phi\partial_t s\in L^2(J;M')$, $0\leqslant\theta(\mathbf{x},t)\leqslant\theta^*(\mathbf{x})$ *a.e.* on $\Omega_T$,

$$B(\mathbf{u},v)=0, \tag{20}$$

$$A(s;\mathbf{u},\mathbf{v})+B(\mathbf{v},p)=(\gamma_1(s),\mathbf{v}), \tag{21}$$

$$\int_0^t(\phi\partial_t s,v)\,\mathrm{d}\tau+\int_0^t(\mathbf{v}_\mathrm{w},\nabla v)\,\mathrm{d}\tau=-\int_0^t(\varphi_2(s),v)_\Gamma\,\mathrm{d}\tau \tag{22}$$

for all $v\in L^\infty(J;W)$, for all $\mathbf{v}\in L^\infty(J;V(0))$, for all $v\in L^2(0,t;M)$, $t\in J$, where $a(s):=(k\lambda(s))^{-1}$ and $\gamma_1(s):=-\gamma_1'(s)/\lambda(s)$.

Under physically reasonable assumptions on the data and the assumption that $\Omega$ is a multiply-connected domain with Lipschitz boundary $\Gamma$, the system (17)–(19) has a weak solution in the weak sense of (20)–(22). Under additional assumptions on the data, it was shown by Chen [15] and [16], that $s$ is Hölder continuous on $\Omega_T$ and the weak solution is unique.

### 2.4. Model problem for convergence analysis

In Heap Leaching it is physically reasonable to assume that a residual saturation $0<s_\mathrm{wr}<1$, an initial saturation $0<s_\mathrm{w}^o<1$ and a saturation of stability $0<s_\mathrm{w}^e<1$ of the leaching solution exist, such that the capillary diffusion coefficient $D_\mathrm{w}$, in (13), satisfies

$$0<D_\mathrm{w}(s_\mathrm{wr})<D_\mathrm{w}(s_\mathrm{w}^o)\leqslant D_\mathrm{w}(s)\leqslant D_\mathrm{w}(s_\mathrm{w}^e)<1,$$

where $0<s_\mathrm{wr}<s_\mathrm{w}^o<s_\mathrm{w}^e<1$. For our convergence analysis we consider the system (11)–(13), under the assumptions that it is not degenerate. In order to simplify our convergence analysis we replace the nonlinear function $D_\mathrm{w}$ in (13) by the constant $\epsilon>0$ defined as

$$\epsilon:=\frac{1}{|\Omega\times J|}\int_{\Omega\times J}D_\mathrm{w}(s(\mathbf{x},t))\,\mathrm{d}\mathbf{x}\,\mathrm{d}t,$$

where $J:=(0,T)$. Additionally, we consider vectorial functions $\mathbf{d}$ and $\mathbf{e}$ such that $\mathbf{d}(s(\mathbf{x},t))\cdot\mathbf{n}=\varphi_1(s(\mathbf{x},t))$, and $\mathbf{e}(s(\mathbf{x},t))\cdot\mathbf{n}=\varphi_2(s(\mathbf{x},t))$, with $\mathbf{x}\in\partial\Omega$, and define the new unknowns $\mathbf{w}$ and $\mathbf{u}_\mathrm{w}$ as

$$\mathbf{w}+\mathbf{d}(s(\mathbf{x},t))=\mathbf{u},$$

$$\mathbf{u}_\mathrm{w}+\mathbf{e}(s(\mathbf{x},t))=\mathbf{v}_\mathrm{w}$$

with $\mathbf{x}\in\Omega$, $t>0$, then, the homogeneous Neumann boundary condition holds for $\mathbf{w}$ and $\mathbf{u}_\mathrm{w}$. Now, we introduce these simplifications in the system (11)–(15) to obtain our Model Problem for the convergence analysis, maintaining the notation $\mathbf{u}$ for the total velocity.

**Definition 1.** Let $\Omega\subset\mathbb{R}^2$ be a convex polygonal bounded domain, $J:=(0,T)$ a time interval and $\Omega_T:=\Omega\times J$. A mapping $(\mathbf{u},p,s):\Omega_T\to\mathbb{R}\times\mathbb{R}\times\mathbb{R}^2$ is called a Strong Solution of the model problem if for all $(\mathbf{x},t)\in\Omega_T$:

$$\nabla\cdot\mathbf{u}=-F(s), \tag{23}$$

$$\mathbf{u}=-k\lambda(\nabla p-\mathbf{G}(s)), \tag{24}$$

$$\phi\frac{\partial s}{\partial t}+\nabla\cdot(\mathbf{f}(s)-\epsilon\nabla s)=Q(s), \tag{25}$$

where $\mathbf{f}(s):=f_\mathrm{w}(s)\mathbf{u}-G_\mathrm{w}(s)\mathbf{g}-\mathbf{e}(s)$, $F(s):=\nabla\cdot\mathbf{d}(s)$, $Q(s):=-\nabla\cdot[f_\mathrm{w}(s)\mathbf{d}(s)+\mathbf{e}(s)]$ and $\mathbf{G}(s):=G_\lambda(s)\mathbf{g}-(k\lambda)^{-1}(s)\mathbf{d}(s)$. The initial conditions are given by

$$s(\mathbf{x},0)=s^o,\quad p(\mathbf{x},0)=p_o \tag{26}$$

for all $\mathbf{x}\in\Omega$ and the boundary conditions are given by

$$\mathbf{u}\cdot\mathbf{n}=0,\quad(\mathbf{f}(s)-\epsilon\nabla s)\cdot\mathbf{n}=0 \tag{27}$$

for all $\mathbf{x}\in\Gamma$ and $t>0$.

## 3. Statement of the discrete problem

### 3.1. Notation and assumptions

We follow a classical notation for unstructured grid for VF and MFE-VF methods used previously in [8,17,18]. Let $\mathscr{T}_h:=\{T_i|$ is a triangle, $i\in I\subset\mathbb{N}\}$ be a unstructured triangulation with fineness $h$ of a bounded domain $\Omega\subset\mathbb{R}^2$. We assume that the following properties are satisfied:

(1) $\Omega=\bigcup_{T\in\mathscr{T}_h}T$.
(2) For $T_i\neq T_j\in\mathscr{T}_h$ one and only one of the following properties hold: $T_i\bigcap T_j=\emptyset$ or $T_i\bigcap T_j=$ common node *of* $T_i,T_j$ or $T_i\bigcap T_j=$ common edge of $T_i,T_j$.
(3) $h:=\sup_{T\in\mathscr{T}_h}\mathrm{diam}(T)<\infty$.
(4) For any angle $\theta$ of a triangle of $\mathscr{T}_h$, one has: $0<\theta<\pi/2$.
(5) There exists $\alpha_1>0$, $\alpha_2>0$ and $h>0$ such that $\forall T\in\mathscr{T}_h$ and for any edge $a$ of the mesh, $\beta_1h^2\leqslant|T|\leqslant\beta_2h^2$, and $\alpha_1h\leqslant l(a)\leqslant\alpha_2h$.

Additionally, we shall use the following notation for the unstructured triangulation:

$|T_i|$: area of $T_i$,
$\mathbf{x}_i$: midpoint of the ambit of $T_i$,
$N_j$: set of neighbour triangles of $T_j$,
$S_{ij}$: joint edge of $T_i$ and $T_j$,
$\mathbf{n}_{ij}$: outward unit normal to $T_i$ in direction $T_j, j\in N_j$,
$\mathscr{A}$: set of all edges of $\mathscr{T}_h$,
$l(a)$: length of edge $a$,
$d(\mathbf{x}_i,S_{ij})$: distance from $\mathbf{x}_i$ to the edge $S_{ij}$.

If $f(\cdot,t)$ is a piecewise continuous function on $\mathscr{T}_h$ and $p$, $\mathbf{u}$, $s$ is a solution of the model problem, we define in addition for $(\mathbf{x},t)\in\Omega_T$:

$f_j=\frac{1}{T_j}\int_{T_j}f(\mathbf{x},t)\,\mathrm{d}\mathbf{x}$.
$s_j(t)$: a constant approximation of $s(\cdot,t)$ on $T_j\in\mathscr{T}_h$.
$\mathbf{n}_a(t)$: unit normal to edge $a\in\mathscr{A}$ at time $t$, such that $\int_a\mathbf{u}(x,t)\cdot\mathbf{n}_a(t)\,\mathrm{d}\sigma\geqslant0$.

$T_a^{\pm}$: the neighbour triangle of $a$, such that $\mathbf{n}_a$ is the outer (inner) normal of $T_a^{\pm}$.
$s_a^+(t)$: the upstream choice of $s(\cdot,t)$ on the edge $a \in \mathscr{A}$.
$d_a := d(\mathbf{x}_a^+, a) + d(\mathbf{x}_a^-, a)$.
$\gamma_a := l(a)/d_a$, $\kappa := \min_{a\in\mathscr{A}}\gamma_a$, and $\Upsilon := \max_{a\in\mathscr{A}}\gamma_a$.
For any variable $\pi$, some times we use $\pi_a^+ = \pi_{T_a^+}$ and $\pi_a^- = \pi_{T_a^-}$ in order to lighten the notation.

Furthermore, we shall use the following notation for the time discretization:

$$J_h := \{t^n \in J | t^n = n\Delta t,\ \text{with } n \in \{0,\dots,M\},\ \text{such that } M\Delta t = T\}$$

$$f^n(\mathbf{x}) := f(\mathbf{x}, t^n), \quad \text{for any function } f(\mathbf{x}, t).$$

For given discrete data $s_i^n$ let the global function $s_h(\cdot, t_n)$ be defined as $s_i^n := s_h(\mathbf{x}, t^n)|_{T_i}$ for all $T_i \in \mathscr{T}_h$ and the global function $s \in L^2(\Omega)$ we define the interpolation $I_h(s)$ as:

$$I_h(s(\cdot,t))|_{T_i} := s(\mathbf{x}_i, t) \quad \text{for all } T_i \in \mathscr{T}_h.$$

The corresponding finite dimensional subspace of $L^2(\Omega)$ is defined as:

$$l^2(\Omega) := \{v \in L^2(\Omega) | v|_T = \text{const},\ \forall T \in \mathscr{T}_h\}$$

with the norm $\|s_h\|^2_{l^2(\Omega)} := \sum_{T_j\in\mathscr{T}_h} T_j s_j^2$.

Finally, let $[s_h]_a := s_a^+ - s_a^-$ be the jump of $s_h$ over an edge $a \in \mathscr{A}$.

**Remark 2.** If $(\mathbf{u}, p, s)$ is a sufficiently smooth solution of the model problem, then we have, for all $T_j \in \mathscr{T}_h$,

$$\phi(\partial_t s)_j + (L(\mathbf{u})s)_j = (Q(s))_j, \tag{28}$$

where $L(\mathbf{u})s := \nabla\cdot(\mathbf{f}(s) - \epsilon\nabla s)$.

### 3.2. The mixed finite element part

Let $\mathbf{V}_h$ and $W_h$ finite dimensional subspaces of $\mathbf{V} := \mathbf{V}(0)$ and $W$, defined as

$$W_h := \{w_h \in W | w_h|_T = \text{const},\ \forall T \in \mathscr{T}_h\}$$

and $\mathbf{V}_h$ is a lowest order Raviart–Thomas space.

**Definition 3.** For fixed $t \in J$ let $s_h(\mathbf{x}, t)$ be given. Then the mixed finite element scheme for the Eqs. (23) and (24) is defined as: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$, such that, for all $(\mathbf{v}_h, \varphi_h) \in \mathbf{V}_h \times W_h$:

$$B(\mathbf{u}_h, \varphi_h) = (F(s_h), \varphi_h), \tag{29}$$
$$A(s_h; \mathbf{u}_h, \mathbf{v}_h) + B(\mathbf{v}_h, p_h) = (\mathbf{G}(s_h), \mathbf{v}_h). \tag{30}$$

### 3.3. The finite volume part

We consider a cell centered finite volume scheme for the Eq. (25) with the IBC (27), in the unknown $s$, *i.e.*, the level of saturation of leaching's solution. For an arbitrary triangle $T_j \in \mathscr{T}_h$

$$\phi\partial_t s + \nabla\cdot(\mathbf{f}(s) - \epsilon\nabla s) = Q(s),$$
$$\phi\frac{1}{T_j}\int_{T_j}\partial_t s\,\mathrm{d}\mathbf{x} + \frac{1}{T_j}\int_{T_j}\nabla\cdot(\mathbf{f}(s) - \epsilon\nabla s)\,\mathrm{d}\mathbf{x} = \frac{1}{T_j}\int_{T_j}Q(s)\,\mathrm{d}\mathbf{x},$$
$$\phi\frac{1}{T_j}\int_{T_j}\partial_t s\,\mathrm{d}\mathbf{x} + \frac{1}{T_j}\sum_{l\in N_j}\int_{S_{jl}}(\mathbf{f}(s) - \epsilon\nabla s)\cdot\mathbf{n}\,\mathrm{d}\boldsymbol{\sigma} = \frac{1}{T_j}\int_{T_j}Q(s)\,\mathrm{d}\mathbf{x}.$$

From this last equality, we can define the discrete relation (see [19])

$$\phi(\partial_t s_h)_j + \frac{1}{T_j}\sum_{l\in N_j}F_{jl} = (Q(s_h))_j,$$

where $F_{jl}(\mathbf{u}, s_h) := g_{jl}(\mathbf{u}; s_{hj}, s_{hl}) - \epsilon\gamma_{jl}(s_{hl} - s_{hj})$, if $S_{jl} \cap \partial\Omega = \emptyset$, $F_{jl}(\mathbf{u}, s_h) := 0$, otherwise, and $g_{jl}(\cdot)$ is a Engquist–Osher numerical flux given by

$$g_{jl}(\mathbf{u}; s_{hj}, s_{hl}) := |S_{jl}|[\Phi_{jl}^+ + \Phi_{jl}^-],$$
$$\Phi_{jl}^+ = \Phi_{jl}(0) + \int_0^{s_{hj}}\max\{\Phi'_{jl}(\xi), 0\}\,\mathrm{d}\xi, \tag{31}$$
$$\Phi_{jl}^- = \int_0^{s_{hl}}\min\{\Phi'_{jl}(\xi), 0\}\,\mathrm{d}\xi$$

with $\Phi_{jl}(s) := \mathbf{f}(s)\cdot\mathbf{n}$. It is well know that the Engquist–Osher numerical flux $g_{jl}(\cdot)$ defined in (31), satisfies [19]: for all $r > 0$, there exists a constant $C = C(r) > 0$ such that for all $u, v, u', v' \in B_r(0)$

$$|g_{jl}(\cdot; u, v) - g_{jl}(\cdot; u', v')| \leqslant C(r)h(|u - u'| + |v - v'|), \tag{32}$$
$$g_{jl}(\cdot; u, v) = -g_{lj}(\cdot; v, u), \tag{33}$$
$$g_{jl}(\cdot; u, u) = |S_{jl}|\mathbf{f}(u)\cdot\mathbf{n}_{jl}. \tag{34}$$

Note that the inequality (32) is a local Lipschitz condition, the identity (33) is the conservation property and the identity (34) is consistency. Finally, the semi discrete finite volume scheme is defined as

**Definition 4.** Let $(\mathbf{u}_h(\mathbf{x}, t), p_h(\mathbf{x}, t)) \in \mathbf{V}_h \times W_h$ for $(\mathbf{x}, t) \in \Omega_T$. Then $s_h(\mathbf{x}, t)$ is defined by the semi discrete finite volume scheme as

$$\phi(\partial_t s_h)_j + (L_h(\mathbf{u}_h)s_h)_j = (Q(s_h))_j, \forall T_j \in \mathscr{T}_h, \tag{35}$$

where $(L_h(\psi)\varsigma)_j := \frac{1}{T_j}\sum_{l\in N_j}F_{jl}(\psi, \varsigma)$ and $s_h(\cdot, 0)|_{T_j} = (s^o(\cdot))_j$. Additionally, the discrete inner product is defined as $(L_h(\psi)\varsigma, \varsigma)_h := \sum_j T_j\varsigma_j(L_h(\psi)\varsigma)_j$.

### 3.4. The combined schemes

#### 3.4.1. The semi discrete scheme

Let $(\mathbf{u}, p, s)$ be a weak solution of (23)–(25). We define the semi discrete combined and decoupled MFE-FE scheme for the model problem as follows:

**Definition 5** (*Coupled*). Find $(\mathbf{u}_h, p_h, s_h) : J \to \mathbf{V}_h \times W_h \times l^2(\Omega)$ with:

(1) $(\mathbf{u}_h, p_h)$ is a solution of the MFE scheme, *i.e.*, for all $(\mathbf{v}_h, \varphi_h) \in \mathbf{V}_h \times W_h$:

$$B(\mathbf{u}_h, \varphi_h) = (F(s_h), \varphi_h), \tag{36}$$

$$A(s_h; \mathbf{u}_h, \mathbf{v}_h) + B(\mathbf{v}_h, p_h) = (\mathbf{G}(s_h), \mathbf{v}_h). \tag{37}$$

(2) $s_h$ is a solution of the semi discrete FV scheme:

$$\phi(\partial_t s_h)_j + (L_h(\mathbf{u}_h)s_h)_j = (Q(s_h))_j \tag{38}$$

with $s_h(\cdot,0)|_{T_j} = (s^o(\cdot))_j$.

**Definition 6** (*Decoupled*). Given $t \in J$, find $(\tilde{\mathbf{u}}(t), \tilde{p}(t), \tilde{s}(t))$ such that:

(1) $(\tilde{\mathbf{u}}(t), \tilde{p}(t)) \in \mathbf{V}_h \times W_h$ is a solution of:

$$B(\tilde{\mathbf{u}}(t), \varphi_h) = (F(s), \varphi_h), \quad \forall \varphi_h \in W_h, \tag{39}$$

$$A(s(t); \tilde{\mathbf{u}}(t), \mathbf{v}_h) + B(\mathbf{v}_h, \tilde{p}(t)) = (\mathbf{G}(s), \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{40}$$

(2) $\tilde{s}(\cdot,t)$ solves:

$$\phi(\partial_t s)_j + (L_h(\mathbf{u})\tilde{s})_j = (Q(s))_j, \tag{41}$$

with $\tilde{s}(\cdot,0)|_{T_j} = (s^o(\cdot))_j$, for all $T_j \in \mathscr{T}_h$.

*3.4.2. The full discrete scheme*

Let $(\mathbf{u}, p, s)$ be a weak solution of (23)–(25). We define the full discrete combined and decoupled MFE-FE scheme for the model problem as follows.

**Definition 7** (*Coupled*). Find $(\mathbf{U}_h, P_h, S_h) : J_h \to \mathbf{V}_h \times W_h \times l^2(\Omega)$ with:

(1) Initial values: $S_j^0 := (s^o)_j$, for all $T_j \in \mathscr{T}_h$.
(2) For $n = 0$ to $M$ do:
  (a) For given $S_h(\cdot, t^n)$ let $(\mathbf{U}_h(\cdot,t^n), P_h(\cdot,t^n)) \in \mathbf{V}_h \times W_h$ be defined as the solution of the MFE scheme, such that, for all $(\mathbf{v}_h, \varphi_h) \in \mathbf{V}_h \times W_h$:

$$B(\mathbf{U}_h, \varphi_h) = (F(S_h), \varphi_h), \tag{42}$$

$$A(S_h; \mathbf{U}_h, \mathbf{v}_h) + B(\mathbf{v}_h, P_h) = (\mathbf{G}(S_h), \mathbf{v}_h). \tag{43}$$

  (b) For given $(\mathbf{U}_h(\cdot,t^n), P_h(\cdot,t^n))$ calculate $S_h(\cdot, t^{n+1})$ with the full discrete FV scheme, defined as:

$$\phi \frac{S_{hj}^{n+1} - S_{hj}^n}{\Delta t} + (L_h(\mathbf{U}_h)S_h)_j = (Q(S_h))_j \tag{44}$$

  for all $T_j \in \mathscr{T}_h$.

**Definition 8** (*Decoupled*). Find $(\tilde{\mathbf{u}}, \tilde{p}, \widetilde{S})$ such that:

1. For each $t^n \in J_h$: $(\tilde{\mathbf{u}}(t^n), \tilde{p}(t^n)) \in \mathbf{V}_h \times W_h$ is a solution of (39) and (40).
2. $\widetilde{S}(\cdot, t^{n+1})$ satisfies the initial condition $\widetilde{S}_j^o = (s^o)_j$ for all $T_j \in \mathscr{T}_h$ and for $n = 0$ to $M$:

$$\phi \frac{\widetilde{S}_j^{n+1} - \widetilde{S}_j^n}{\Delta t} + (L_h(\mathbf{u})\widetilde{S})_j = (Q(s))_j, \tag{45}$$

where $\widetilde{S}_j^{n+1} := \widetilde{S}(t^{n+1})|_{T_j}$ for all $T_j \in \mathscr{T}_h$.

## 4. Main results

*4.1. Convergence of the semi discrete scheme*

**Theorem 9.** *Let $(\mathbf{u}, p, s)$ be a weak solution of (23)–(25) and $(\mathbf{u}_h, p_h, s_h)$ a solution of (36)–(38). Then, there exist constants $K_1, K_2 > 0$, depending on some higher order Sobolev norms of $(\mathbf{u}, p, s)$, but independent of $h$ and $\epsilon$, such that:*

$$\|\mathbf{u} - \mathbf{u}_h\|^2_{L^\infty(J;V)} + \|p - p_h\|^2_{L^\infty(J;W)} + \|s - s_h\|^2_{L^\infty(J;L^2(\Omega))} + \epsilon\kappa \int_J \sum_{a\in\mathscr{A}} [I_h(s)(t) - s_h(t)]_a^2 \,\mathrm{d}t \leqslant h^2 K_1(1 + \exp(K_2 T)).$$

This theorem is proved at the end of Section 6, after some previous results.

*4.2. Convergence of the full discrete scheme*

**Theorem 10.** *Let $(\mathbf{u}, p, s)$ be weak solution of (23)–(25) and $(\mathbf{U}_h, P_h, S_h)$ solution of (42)–(44). If $\Delta t$ satisfies the CFL condition*

$$\frac{\Delta t}{h^2} \leqslant \phi \frac{\kappa}{4\epsilon\Upsilon^2} \tag{46}$$

*then there exists a constant $K_i > 0$, $i = 3,4,5,6$, depending on some higher order Sobolev norms of the exact solution, but independent of $h$ and $\epsilon$ such that:*

$$\|\mathbf{u}(t^n) - \mathbf{U}_h^n\|_V^2 + \|p(t^n) - P_h^n\|_W^2 + \|s(t^n) - S_h^n\|^2_{L^2(\Omega)}$$
$$\leqslant K_3 h^2 + K_4 \left\{ \left[ \frac{h^2}{(\epsilon\kappa)^2} + T\left(\frac{T}{M\phi^2} + \frac{h^2\phi}{\epsilon\kappa}\right) + \frac{T}{\phi}\left(\frac{T}{M\phi} + \frac{h^2}{\epsilon\kappa}\right)\right] \|\mathbf{f}\|^2_{L^\infty(\Omega)} + \left(\frac{h}{\epsilon\kappa}\right)^2 \|Q(s) - \phi\partial_t s\|^2_{L^\infty(\Omega)} + \frac{h^2\Upsilon^2}{\epsilon\kappa^2}\mathscr{D}(s) + \frac{T}{2}\left(\frac{T}{M} + \frac{\phi}{\kappa\epsilon}\right)(\mathscr{E}^* + K_5 h^2) \right\} + K_6 \|Q\|^2_{L^\infty(\Omega)}$$

*with $M\Delta t = T$.*

This theorem is proved at the end of Section 7, after some previous results.

## 5. Preliminary results

We have the following estimates from the geometric properties of an unstructured grid [8,17,18]:

**Lemma 11.** *Let $\mathscr{T}_h$ be a unstructured triangulation, $T_j, T_l \in \mathscr{T}_h$, $\mathbf{x}, \mathbf{y} \in T_j \cup T_l$, $\varsigma, \delta \in l^2(\Omega)$, $\omega \in L^2(J; H^2(\Omega))$, $\eta \in H^2(\Omega)$. Then there exist constants $C_1, C_2, C_3 > 0$, independent of $h$, such that:*

$$|\omega(\mathbf{x}) - \omega(\mathbf{y})| \leqslant C_1 \|\omega\|_{H^2(T_j \cup T_l)}, \tag{47}$$

$$C_2 \|\delta\|^2_{l^2(\Omega)} \leqslant \sum_{a\in\mathscr{A}} [\delta]_a^2 \leqslant \frac{2}{h^2}\|\delta\|^2_{l^2(\Omega)}, \tag{48}$$

$$\|\varsigma - \delta\|_{L^2(\Omega)} = \|\varsigma - \delta\|_{l^2(\Omega)}, \tag{49}$$

$$\|\eta - \delta\|^2_{L^2(\Omega)} \leqslant C_3 h^2 \|\eta\|^2_{H^2(\Omega)} + \|I_h(\eta) - \delta\|^2_{l^2(\Omega)}. \tag{50}$$

On the other hand, we have the following estimates for the numerical flux:

**Lemma 12.** *Let $\psi \in \mathbf{V}$ be a given vector. Then there exists constants $C_4, C_5 > 0$, independent of $h$ and $\epsilon$, such that*:

$$\left[\sum_{a\in\mathscr{A}} g_a^2(\psi;\varsigma_a^+,\varsigma_a^-)\right]^{1/2} \leqslant C_4 h\|\mathbf{f}\|_{L^\infty(\Omega)}, \tag{51}$$

$$\left[\sum_{a\in\mathscr{A}} (\partial_t g_a(\psi;\varsigma_a^+,\varsigma_a^-))^2\right]^{1/2} \leqslant C_5 h\|\mathbf{f}'\|_{L^\infty(\Omega)}\|\partial_t\varsigma\|_{L^\infty(\Omega)}. \tag{52}$$

**Proof.** By definition is enough to see that

$$\begin{aligned} g_a(\psi;\varsigma_a^+,\varsigma_a^-) &\leqslant |S_a||\Phi_{jl}(\max\{\varsigma_a^+,\varsigma_a^-\})| \\ &\leqslant \alpha_2 h|\mathbf{f}(\max\{\varsigma_a^+,\varsigma_a^-\})|. \end{aligned}$$

and then we obtain (51). On the other hand, by definition and the application of the Leibnitz's rule, we have

$$\begin{aligned} \partial_t g_a(\psi;\eta,\tau) &= |S_a|[\partial_t\Phi_a^+(\eta) + \partial_t\Phi_a^-(\tau)] \\ &= |S_a|\left[\partial_t\int_0^{\eta(t)} \max\{\Phi_a'(\delta),0\}\,\mathrm{d}\delta \right. \\ &\quad \left. + \partial_t\int_0^{\tau(t)} \min\{\Phi_a'(\delta),0\}\,\mathrm{d}\delta\right] \\ &\leqslant |S_a|\max\{|\partial_t\eta|,|\partial_t\tau|\}\sup_\xi|\Phi_a'(\xi)|. \end{aligned}$$

and then we obtain (52). $\square$

Finally, from (35) and the Hölder's inequality we obtain the following estimates for the operator $L_h$:

**Lemma 13.** *For the discrete inner product $(L_h(\psi)\varsigma,\varsigma)_h$ we have*

(1) *Coerciveness*:

$$(L_h(\psi)\varsigma,\varsigma)_h \geqslant \epsilon\kappa\sum_{a\in\mathscr{A}}[\varsigma]_a^2 + \sum_{a\in\mathscr{A}}[\varsigma]_a g_a(\psi;\varsigma_a^+,\varsigma_a^-).$$

(2) *Boundedness*:

$$(L_h(\psi)\varsigma,\delta)_h \leqslant \left[\epsilon\Upsilon\left(\sum_{a\in\mathscr{A}}[\varsigma]_a^2\right)^{1/2} + \left(\sum_{a\in\mathscr{A}} g_a^2(\psi;\varsigma_a^+,\varsigma_a^-)\right)^{1/2}\right] \times \left(\sum_{a\in\mathscr{A}}[\delta]_a^2\right)^{1/2}.$$

## 6. Convergence of the semi discrete scheme

**Theorem 14.** *Let $(\mathbf{u},p,c)$ be the weak solution of (23)–(25). If $p(\tau)\in H^1(\Omega)$, $\mathbf{u}(\tau)\in(H^1(\Omega))^2$ and $\operatorname{div}\mathbf{u}(\tau)\in H^1(\Omega)$ for any fixed time $\tau\in J$, then the scheme (39), (40) has a unique solution $(\tilde{\mathbf{u}}(\tau),\tilde{p}(\tau))\in\mathbf{V}_h\times W_h$ and there exists a constant $K_7>0$, independent of $h$ and $s(\tau)$, such that*:

$$\begin{aligned} &\|(\mathbf{u}-\tilde{\mathbf{u}})(\tau)\|_{H(\mathrm{div};\Omega)} + \|(p-\tilde{p})(\tau)\|_{L^2(\Omega)} \\ &\quad\leqslant K_7 h(|p(\tau)|_{H^1(\Omega)} + |\mathbf{u}(\tau)|_{(H^1(\Omega))^2} + |\operatorname{div}\mathbf{u}(\tau)|_{H^1(\Omega)}). \end{aligned} \tag{53}$$

**Proof.** Clearly, the bilinear form $A(s;\cdot,\cdot)$ is coercive and $B(\cdot,\cdot)$ satisfies the inf–sup condition. Then, using [20, Theorem 1.1] we have that the scheme (39), (40) has a unique solution $(\tilde{\mathbf{u}}(\tau),\tilde{p}(\tau))\in\mathbf{V}_h\times W_h$ and there exists a constant $C_6>0$ such that

$$\begin{aligned} &\|(\mathbf{u}-\tilde{\mathbf{u}})(\tau)\|_{H(\mathrm{div};\Omega)} + \|(p-\tilde{p})(\tau)\|_{L^2(\Omega)} \\ &\quad\leqslant C_6\left[\inf_{v_h\in V_h}\|\mathbf{u}(\tau)-\mathbf{v}_h\|_{H(\mathrm{div};\Omega)} + \inf_{w_h\in W_h}\|p(\tau)-w_h\|_{L^2(\Omega)}\right]. \end{aligned}$$

Now, using standard approximation properties to estimate the right hand side expression of this last inequality (see [12]), we obtain (53). $\square$

**Theorem 15.** *Let $\tilde{s}$ be the solution of (41) and $(\mathbf{u},p,s)$ the weak solution of (23)–(25). Let $e_h$ be defined as $e_h:=\tilde{s}-I_h(s)$. Then, for all $t\in J$ there exists a constant $K_8>0$, independent of $h$ and $\epsilon$, such that the following estimate holds*:

$$\begin{aligned} \frac{\epsilon\kappa}{2}\sum_a[e_h(t)]_a^2 &\leqslant 3K_8\frac{h^2}{\epsilon\kappa}\Big\{\|\mathbf{f}\|^2_{L^\infty(\Omega)} + \|Q(s)-\phi\partial_t s\|^2_{L^\infty(\Omega)} \\ &\quad + \epsilon\Upsilon^2\|\mathscr{D}(s)\|^2_{L^\infty(\Omega)}\Big\}(t), \end{aligned} \tag{54}$$

*where* $\mathscr{D}(s):=\sum_{0\leqslant|\alpha|\leqslant 2}|D^\alpha s|^2$.

**Proof.** for $e_h:=\tilde{s}-I_h(s)$ we have

$$\begin{aligned} (L_h(\mathbf{u})e_h,e_h)_h &= -\epsilon\sum_a[e_h]_a[I_h(s)]_a\gamma_a + \sum_a[e_h]_a\widetilde{G}_a \\ &\quad + \sum_j T_j e_j[(Q(s))_j - \phi(\partial_t s)_j], \end{aligned} \tag{55}$$

where $\widetilde{G}_a := g_a(\mathbf{u};e_j,e_l) - g_a(\mathbf{u};\tilde{s}_j,\tilde{s}_l)$. Using the coerciveness property of $L_h$ from Lemma 13, we can estimate

$$\epsilon\kappa\sum_a[e_h]_a^2 \leqslant t_1+t_2+t_3, \tag{56}$$

with

$$t_1 := -\epsilon\sum_a[e_h]_a[I_h(s)]_a\gamma_a,\quad t_2 := -\sum_a[e_h]_a g_a(\mathbf{u};\tilde{s}_j,\tilde{s}_l),$$

$$t_3 := \sum_j T_j e_j[(Q(s))_j - \phi(\partial_t s)_j].$$

About $t_1$, using the Hölder and Young inequalities, we obtain for each $\theta_1>0$ that $t_1\leqslant\frac{\epsilon}{2}\{\theta_1\sum_a[e_h]_a^2 + (\Upsilon^2/\theta_1)\sum_a[I_h(s)]_a^2\}$, but by (47) and (48), there exist constants $C_7,C_8>0$ such that

$$\begin{aligned} [I_h(s)]_a^2 &= |s(\mathbf{x}_a^+,t)-s(\mathbf{x}_a^-,t)|^2 \leqslant C_7\|s(\cdot,t)\|^2_{H^2(T_a^+\cup T_a^-)} \\ &= C_7\sum_{0\leqslant|\alpha|\leqslant 2}\|D^\alpha s(\cdot,t)\|^2_{L^2(T_a^+\cup T_a^-)} \\ &\leqslant C_8 h^2\sup_{T_a^+\cup T_a^-}|\mathscr{D}^2(s(\mathbf{x},t))|, \end{aligned}$$

where $\mathscr{D}^2(s(\mathbf{x},t)):=\sum_{0\leqslant|\alpha|\leqslant 2}|D^\alpha s(\mathbf{x},t)|^2$. Therefore, there exists $C_9>0$ such that

$$t_1 \leqslant \frac{\epsilon}{2}\left\{\theta_1 \sum_a [e_h]_a^2 + \frac{\Upsilon^2}{\theta_1} C_9 h^2 \|\mathscr{D}(s)\|^2_{L^\infty(\Omega)}\right\}. \tag{57}$$

About $t_2$, using the Hölder and Young inequalities and Lemma 12, we obtain for each $\theta_2 > 0$, that there exists a constant $C_{10} > 0$, such that

$$t_2 \leqslant \frac{1}{2}\left\{\theta_2 \sum_a [e_h]_a^2 + \frac{1}{\theta_2} C_{10} h^2 \|\mathbf{f}\|^2_{L^\infty(\Omega)}\right\}. \tag{58}$$

About $t_3$, using Hölder and Young inequalities and the inequality (48), we obtain for each $\theta_3 > 0$, that there exist constants $C_{11}, C_{12} > 0$ such that

$$\begin{aligned} t_3 &\leqslant \left(\sum_j T_j e_j^2\right)^{1/2} \left(\sum_j T_j \Big[(Q(s))_j - \phi(\partial_t s)_j\Big]^2\right)^{1/2} \\ &\leqslant \frac{1}{2}\left[\theta_3 \|e_h\|^2_{l^2(\Omega)} + \frac{1}{\theta_3}\Big(C_{11} h^2 \|Q(s) - \phi\partial_t s\|^2_{L^\infty(\Omega)}\Big)\right] \\ &\leqslant \frac{1}{2}\left[(\theta_3/C_{12}) \sum_a [e_h]_a^2 + \frac{1}{\theta_3}\Big(C_{11} h^2 \|Q(s) - \phi\partial_t s\|^2_{L^\infty(\Omega)}\Big)\right]. \end{aligned} \tag{59}$$

Finally, with $\theta_1 = \kappa/3$, $\theta_2 = \epsilon\kappa/3$ and $\theta_3 = (\epsilon\kappa/3)C_{12}$, and replacing (56)–(59) in (55), we get (54). □

**Theorem 16.** *Let $\tilde{s}$ be the solution of (41) and $(\mathbf{u}, p, s)$ the weak solution of (23)–(25). Let $e_h$ be defined as $e_h := \tilde{s} - I_h(s)$. Then there exists a constant $K_9 > 0$, independent of h and $\epsilon$, such that the following estimate holds, for all $t \in J$*

$$\frac{\epsilon\kappa}{4} \sum_a [\partial_t e_h]_a^2 \leqslant \frac{2K_9 h^2}{\epsilon\kappa}\Big(\epsilon\Upsilon^2 \widetilde{\mathscr{D}}^2(\partial_t s) + \|\mathbf{f}'\|^2_{L^\infty(\Omega)} + \|\partial_t \tilde{s}\|^2_{L^\infty(\Omega)} + \|\partial_t Q(s) - \phi\partial_{tt} s\|^2_{L^\infty(\Omega)}\Big),$$

*where* $\widetilde{\mathscr{D}}(\partial_t s) := \|\sum_{|\alpha|\leqslant 2} |D^\alpha(\partial_t s(t))|^2\|_{L^\infty(\Omega)}$.

**Proof.** First we will prove the following identity:

$$\begin{aligned} (L_h(\mathbf{u})\partial_t e_h, \partial_t e_h)_h &= \sum_{a\in\mathscr{A}} [\partial_t e_h]_a g_a(\mathbf{u}; \partial_t e_a^+, \partial_t e_a^-) \\ &\quad + \epsilon \sum_{a\in\mathscr{A}} [\partial_t e_h]_a \partial_t (s_a^- - s_a^+)\gamma_a \\ &\quad - \sum_{a\in\mathscr{A}} [\partial_t e_h]_a \partial_t g_a(\mathbf{u}; \tilde{s}_a^+, \tilde{s}_a^-) \\ &\quad + \sum_j |T_j| \partial_t e_j \Big[-\phi\partial_t(\partial_t s)_j + \partial_t(Q(s))_j\Big]. \end{aligned} \tag{60}$$

Then, we apply $\partial_t$ in both sides of the semi-discrete scheme (41) to obtain:

$$-\frac{\epsilon}{|T_j|} \sum_{l\in N_j} (\partial_t \tilde{s}_l - \partial_t \tilde{s}_j)\gamma_{jl} + \frac{1}{|T_j|} \sum_{l\in N_j} \partial_t g_{jl}(\mathbf{u}; \tilde{s}_j, \tilde{s}_l) = -\phi\partial_t(\partial_t s)_j + \partial_t(Q(s))_j.$$

Then, using the identity $\partial_t \tilde{s}_l - \partial_t \tilde{s}_j = \partial_t e_l - \partial_t e_j + \partial_t(s_l - s_j)$, we deduce

$$(L_h(\mathbf{u})\partial_t e_h)_j - \frac{1}{|T_j|} \sum_l g_{jl}(\mathbf{u}; \partial_t e_j, \partial_t e_l) - \frac{\epsilon}{|T_j|} \sum_l \partial_t (s_l - s_j)\gamma_{jl} + \frac{1}{|T_j|} \sum_l \partial_t g_{jl}(\mathbf{u}; \tilde{s}_j, \tilde{s}_l) = -\phi\partial_t(\partial_t s)_j + \partial_t(Q(s))_j.$$

Multiplying this last equality by $|T_j|\partial_t e_j$ and summing up over all triangles $T_j \in \mathscr{T}_h$ we obtain

$$\begin{aligned} (L_h(\mathbf{u})\partial_t e_h, \partial_t e_h)_h &= \sum_{j,l} \partial_t e_j g_{jl}(\mathbf{u}; \partial_t e_j, \partial_t e_l) + \epsilon \sum_{jl} \partial_t e_j \partial_t (s_l - s_j)\gamma_{jl} \\ &\quad - \sum_{jl} \partial_t e_j \partial_t g_{jl}(\mathbf{u}; \tilde{s}_j, \tilde{s}_l) \\ &\quad + \sum_j |T_j| \partial_t e_j \Big[-\phi\partial_t(\partial_t s)_j + \partial_t(Q(s))_j\Big]. \end{aligned}$$

Finally, applying $\sum_{jl} A_{jl} = \sum_{a\in\mathscr{A}} [A_{T_a^+} + A_{T_a^-}]$ we obtain (60).

Now, using the first inequality of Lemma 13, we get:

$$\epsilon\kappa \sum_a [\partial_t e_h]_a^2 \leqslant t_4 + t_5 + t_6, \tag{61}$$

with

$$t_4 := -\epsilon \sum_{a\in\mathscr{A}} [\partial_t e_h]_a [\partial_t I_h(s)]_a \gamma_a, \quad t_5 := -\sum_{a\in\mathscr{A}} [\partial_t e_h]_a \partial_t g_a(\mathbf{u}; \tilde{s}_a^+, \tilde{s}_a^-),$$

$$t_6 := \sum_j |T_j| \partial_t e_j [-\phi\partial_t(\partial_t s)_j + \partial_t(Q(s))_j].$$

To obtain bounds for $t_4$, $t_5$ and $t_6$ we follows the main ideas of Theorem 15. For each $\theta_1, \theta_2, \theta_3 > 0$, there exist constants $C_i > 0$, $i = 13, 14, 15, 16$, such that

$$t_4 \leqslant \frac{\epsilon}{2}\left\{\theta_1 \sum_a [\partial_t e_h]_a^2 + \frac{\Upsilon^2}{\theta_1}(C_{13} h^2)\widetilde{\mathscr{D}}(\partial_t s)\right\},$$

$$t_5 \leqslant \frac{1}{2}\left\{\theta_2 \sum_a [\partial_t e_h]_a^2 + \frac{1}{\theta_2} C_{14} h^2 \|\mathbf{f}'\|^2_{L^\infty(\Omega)} \|\partial_t \tilde{s}\|^2_{L^\infty(\Omega)}\right\}.$$

$$t_6 \leqslant \frac{1}{2}\left\{(\theta_3/C_{16}) \sum_a [\partial_t e_h]_a^2 + \frac{1}{\theta_3} C_{15} h^2 \|\partial_t Q(s) - \phi\partial_{tt} s\|^2_\infty\right\},$$

where $\widetilde{\mathscr{D}}(\partial_t s) := \|\sum_{|\alpha|\leqslant 2} |D^\alpha(\partial_t s(t))|^2\|_{L^\infty(\Omega)}$. Finally, it is sufficient to choose $\theta_1 = \kappa/2$, $\theta_2 = \epsilon\kappa/2$, and $\theta_3 = C_{16}\epsilon\kappa/2$. □

Now we use some results of Ohlberger [8], which established the stability of the decoupled and nonlinear semi discrete schemes (see [8, Lemma 5.12 and 5.13]).

**Lemma 17.** *Let $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{s})$ be the solution of (39)–(41), and let $(\mathbf{u}_h, p_h, s_h)$ be the solution of (36)–(38). Then there exist constants $C_{17}, C_{18}, C_{19} > 0$ (with $C_{18} = C_{18}(\epsilon)$), independent of h, such that for all $t \in J$ the following estimates hold:*

$$\|\tilde{\mathbf{u}}(t)\|_{L^\infty(\Omega)} + \|\tilde{p}(t)\|_{L^\infty(\Omega)} \leqslant C_{17}, \tag{62}$$

$$\|\tilde{s}(t)\|_{L^\infty(\Omega)} \leqslant C_{18}, \tag{63}$$

$$h\|\mathbf{u}_h(t)\|_{L^\infty(\Omega)} \leqslant C_{19}\Big(h + \|\mathbf{u}(t) - \mathbf{u}_h(t)\|_{L^2(\Omega)}\Big). \tag{64}$$

**Theorem 18.** *Let $(\mathbf{u}_h, p_h, s_h)$ be the solution of (36)–(38), $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{s})$ the solution of (39)–(41) and $(\mathbf{u}, p, s)$ the weak solution of (23)–(25). Then, exists a constant $K_{10} > 0$, independent of $h$ and $\epsilon$, such that for all $\tau \in J$ the following estimate holds*

$$\|(\mathbf{u}_h - \tilde{\mathbf{u}})(\tau)\|_{H(\mathrm{div};\Omega)} + \|(p_h - \tilde{p})(\tau)\|_{L^2(\Omega)} \leqslant K_{10}\Big(1 + \|\tilde{\mathbf{u}}(\tau)\|_{L^\infty(\Omega)}\Big)\|(s - s_h)(\tau)\|_{L^2(\Omega)}.$$

**Proof.** Subtracting (29), (30) from (39), (40) we get

$$B(\mathbf{u}_h^*, \varphi_h) = (F(s_h) - F(s), \varphi_h),$$
$$A(s_h; \mathbf{u}_h^*, \mathbf{v}_h) + B(\mathbf{v}_h, p_h^*) = (\mathbf{G}(s_h) - \mathbf{G}(s), \mathbf{v}_h) + A(s; \tilde{\mathbf{u}}, \mathbf{v}_h) - A(s_h; \tilde{\mathbf{u}}, \mathbf{v}_h),$$

which is a discrete saddle point problem in $(\mathbf{u}_h^*, p_h^*) := (\mathbf{u}_h - \tilde{\mathbf{u}}, p_h - \tilde{p})$. Finally, the theorem follows from [20, Remark 1.3] and the Lipschitz continuity of $F(\cdot)$, $\mathbf{G}(\cdot)$ and $a(\cdot)$. $\square$

**Theorem 19.** *Let $\tilde{s}$ be the solution of (41) and let $(\mathbf{u}_h, p_h, s_h)$ be the solution of (36)–(38). Then, exists constants $K_{11} > 0$ and $K_{12} > 0$, independent of $h$ and $\epsilon$, such that*

$$\epsilon\kappa\int_0^T \sum_a [e_h]_a^2 + \|e_h(t)\|_{l^2(\Omega)}^2 \leqslant (TK_{11}h^2)\exp(2K_{12}T), \tag{65}$$

*where $e_h := \tilde{s} - s_h$.*

**Proof.** Subtracting Eq. (35) from Eq. (41), we obtain

$$(L_h(\mathbf{u}_h)e_h)_j + \frac{1}{|T_j|} \times \sum_l \big[g_{jl}(\mathbf{u}; \tilde{s}_j, \tilde{s}_l) - g_{jl}(\mathbf{u}_h; s_{hj}, s_{hl}) - g_{jl}(\mathbf{u}_h; e_j, e_l)\big] = \phi\Big(\partial_t(s_h - s)\big)_j + (Q(s) - Q(s_h)\Big)_j.$$

Multiplying this last equation by $|T_j|e_j$ and summing over all $T_j \in \mathscr{T}_h$ yields after substraction of $(\partial_t\tilde{s})_j$

$$(L_h(\mathbf{u}_h)e_h, e_h)_h + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_j |T_j|e_j^2 + \sum_{a\in\mathscr{A}}[e_h]_a G_a = \phi\sum_j |T_j|e_j(\partial_t(\tilde{s} - s))_j + \sum_j |T_j|e_j(Q(s) - Q(s_h))_j,$$

where $G_a := g_a(\mathbf{u}; \tilde{s}_j, \tilde{s}_l) - g_a(\mathbf{u}_h; s_{hj}, s_{hl}) - g_a(\mathbf{u}_h; e_j, e_l)$, with $j \equiv T_a^+$ and $l \equiv T_a^-$. On the other hand by the first inequality of Lemma 13 we obtain

$$\epsilon\kappa\sum_a [e_h]_a^2 + \frac{1}{2}\frac{d}{dt}\sum_j e_j^2 \leqslant t_7 + t_8 + t_9,$$

with

$$t_7 := \sum_a [e_h]_a(g_a(\mathbf{u}_h; s_{hj}, s_{hl}) - g_a(\mathbf{u}; \tilde{s}_j, \tilde{s}_l)),$$
$$t_8 := \phi\sum_j T_j e_j(\partial_t(\tilde{s} - s))_j$$
$$t_9 := \sum_j T_j e_j(Q(s) - Q(s_h))_j.$$

To obtain bounds for $t_7$, $t_8$ and $t_9$ we follows the main ideas of Theorem 15. From Lemma 11 and Theorem 16, we have that, for each $\theta_1, \theta_2, \theta_3 > 0$, there exists constants $C_i > 0, i = 20, 21, 22, 23, 24$, such that

$$t_7 \leqslant \frac{1}{2}\left\{\theta_1\sum_a [e_h]_a^2 + \frac{1}{\theta_1}C_{20}h^2\|\mathbf{f}\|_{L^\infty(\Omega)}^2\right\},$$
$$\begin{aligned} t_8 &\leqslant \phi\left(\sum_j T_j e_j^2\right)^{1/2}\left(\sum_j T_j(\partial_t(\tilde{s} - s))_j^2\right)^{1/2} \\ &\leqslant \phi\frac{1}{2}\left\{\theta_2\|e_h\|_{l^2(\Omega)}^2 + \theta_2^{-1}\|\partial_t(\tilde{s} - s)\|_{L^2(\Omega)}^2\right\} \\ &\leqslant \phi\frac{1}{2}\left\{\theta_2\|e_h\|_{l^2(\Omega)}^2 + \theta_2^{-1}[C_{21}h^2\|\partial_t s\|_{H^2(\Omega)}^2 + C_{22}\sum_a \left[\partial_t I_h(s) - \partial_t\tilde{s}\right]_a^2\right\} \\ &\leqslant \phi\frac{1}{2}\left\{\theta_2\|e_h\|_{l^2(\Omega)}^2 + \theta_2^{-1}\left[C_{21}h^2\|\partial_t s\|_{H^2(\Omega)}^2 + C_{23}h^2\right]\right\}, \end{aligned}$$
$$t_9 \leqslant \frac{1}{2}\left\{\theta_3\|e_h\|_{l^2(\Omega)}^2 + \frac{1}{\theta_3}C_{24}h^2\|Q\|_{L^\infty(\Omega)}^2\right\}.$$

Therefore, with $\theta_1 = \epsilon\kappa$, we get that there exist $K_{11}, K_{12} > 0$ such that

$$\frac{\epsilon\kappa}{2}\sum_a [e_h]_a^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_h(t)\|_{l^2(\Omega)}^2 \leqslant K_{11}\|e_h(t)\|_{l^2(\Omega)}^2 + K_{12}h^2.$$

After integration with respect to time we get the statement of the proof by applying the Gronwall's Lemma and using $e_h(0) \equiv 0$

$$\frac{\epsilon\kappa}{2}\int_0^t \sum_a [e_h]_a^2 + \frac{1}{2}\|e_h(t)\|_{l^2(\Omega)}^2 \leqslant T(K_{11}h^2)\exp(2K_{12}T). \qquad \square$$

**Proof of the Theorem 9.** Applying the triangle inequality, Theorems 14 and 18 we get:

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}_h\|_{H(\mathrm{div};\Omega)}(t) + \|p - p_h\|_{L^2(\Omega)}(t) \\ &\quad \leqslant \|\mathbf{u} - \tilde{\mathbf{u}}\|_{H(\mathrm{div};\Omega)}(t) + \|\tilde{\mathbf{u}} - \mathbf{u}_h\|_{H(\mathrm{div};\Omega)}(t) + \|p - \tilde{p}\|_{L^2(\Omega)}(t) \\ &\qquad + \|\tilde{p} - p_h\|_{L^2(\Omega)}(t) \\ &\quad \leqslant C_{25}h + C_{26}\|s - s_h\|_{L^2(\Omega)}(t). \end{aligned}$$

On the other hand, applying Lemma 11, Theorems 15 and 19, we obtain

$$
\begin{aligned}
&\|s-s_h\|^2_{L^2(\Omega)}(t)\\
&\quad\leqslant 2\Big(\|s-I_h(s)\|^2_{L^2(\Omega)}(t)+\|I_h(s)-s_h\|^2_{L^2(\Omega)}(t)\Big)\\
&\quad\leqslant 2\Big\{\Big(C_{27}h^2\|s\|^2_{H^2(\Omega)}+\|I_h(s)-I_h(s)\|^2_{l^2(\Omega)}\Big)\\
&\qquad+\Big(\|I_h(s)-\tilde{s}\|^2_{l^2(\Omega)}+\|\tilde{s}-s_h\|^2_{l^2(\Omega)}\Big)\Big\}\\
&\quad\leqslant 2\left\{C_{27}h^2\|s\|^2_{H^2(\Omega)}+\left(\sum_a[I_h(s)-\tilde{s}]^2_a+\|\tilde{s}-s_h\|^2_{l^2(\Omega)}\right)\right\}\\
&\quad\leqslant 2\Big\{C_{27}h^2\|s\|^2_{H^2(\Omega)}+\big[C_{28}h^2+(TK_{11}h^2)\exp(2K_{12}T)\big]\Big\}\\
&\quad\leqslant C_{29}h^2+(2TK_{11}h^2)\exp(2K_{12}T).
\end{aligned}
$$

Since this holds for all $t\in J$, we have proven the $L^\infty$-estimate in time. Finally, with Gronwall's Lemma, Theorems 15, and 19, we have

$$
\begin{aligned}
&\epsilon\kappa\int_J\sum_{a\in\mathscr{A}}[I_h(s)(t)-s_h(t)]^2_a\mathrm{d}t\\
&\quad\leqslant 2\left[\epsilon\kappa\int_J\sum_{a\in\mathscr{A}}[\tilde{s}(t)-s_h(t)]^2_a\mathrm{d}t+\epsilon\kappa\int_J\sum_{a\in\mathscr{A}}[I_h(s)(t)-\tilde{s}(t)]^2_a\mathrm{d}t\right]\\
&\quad\leqslant C_{30}h^2+(2TK_{11}h^2)\exp(2K_{12}T).\qquad\square
\end{aligned}
$$

## 7. Convergence of the full discrete scheme

**Lemma 20.** *Let* $(\mathbf{u},p,s)$ *be the weak solution of (23)–(25),* $\tilde{s}$ *the solution of (41) and* $\widetilde{S}$ *the solution of (45). Define moreover* $e^n_h:=\widetilde{S}^n-\tilde{s}(t^n),\ 0\leqslant n\leqslant M$. *Then, there exist constants* $C_i>0,\ i=31,32,33$, *independent of* $h$ *and* $\epsilon$, *such that*:

$$
\begin{aligned}
&\|e^{n+1}_h-e^n_h\|_{l^2(\Omega)}\\
&\quad\leqslant\frac{\sqrt{2}\Delta t}{h\phi}\left[\epsilon\Upsilon\left(\sum_a[e^n_h]^2_a\right)^{1/2}+C_{31}h\|\mathbf{f}\|_{L^\infty(\Omega)}\right]\\
&\qquad+\Delta t\left\{\int_{t^n}^{t^{n+1}}\left\|\frac{\partial^2 s}{\partial t^2}(s)\right\|_{L^2(\Omega)}\mathrm{d}s+C_{32}h\Big[1+\|\partial_t s\|_{L^\infty(J;H^2(\Omega))}\Big]\right\}\\
&\qquad+C_{33}\frac{\sqrt{2}\Delta t}{\phi}\|\mathbf{f}\|_{L^\infty(\Omega)}.
\end{aligned}
$$

**Proof.** Let $0\leqslant n\leqslant M$. Subtracting Eq. (41) from Eq. (45) we have

$$
\begin{aligned}
&e^{n+1}_j-e^n_j+\frac{\Delta t}{\phi}\Big[(L_h(\mathbf{u}(t^n))\widetilde{S}^n)_j-(L_h(\mathbf{u})\tilde{s}^n)_j(t^n)\Big]\\
&\quad=\Delta t(\partial_t s)_j(t^n)-\tilde{s}_j(t^{n+1})+\tilde{s}_j(t^n).
\end{aligned}
$$

Thus, replacing in this last equation the identity $(L_h(\mathbf{u}(t^n))\widetilde{S}^n)_j-(L_h(\mathbf{u})\tilde{s}^n)_j(t^n)=(L_h(\mathbf{u}(t^n))e_h)_j+\frac{1}{|T_j|}\sum G_{jl}$, where

$$
G_{jl}:=g_{jl}(\mathbf{u}(t^n);\widetilde{S}^n_j,\widetilde{S}^n_l)-g_{jl}(\mathbf{u}(t^n);e^n_j,e^n_l)-g_{jl}(\mathbf{u}(t^n);\tilde{s}^n_j,\tilde{s}^n_l),\tag{66}
$$

we obtain

$$
\begin{aligned}
e^{n+1}_j-e^n_j=&-\frac{\Delta t}{\phi}\left[(L_h(\mathbf{u}(t^n))e_h)_j+\frac{1}{|T_j|}\sum G_{jl}\right]\\
&+\Delta t(\partial_t s)_j(t^n)-\tilde{s}_j(t^{n+1})+\tilde{s}_j(t^n).
\end{aligned}\tag{67}
$$

Multiplying by $(e^{n+1}_j-e^n_j)|T_j|$ and summing up over $j$ this yields:

$$
\begin{aligned}
&\|e^{n+1}_h-e^n_h\|^2_{l^2(\Omega)}\\
&\quad\leqslant\frac{\Delta t}{\phi}|(L_h(\mathbf{u}(t^n))e^n_h,e^{n+1}_h-e^n_h)_h|\\
&\qquad+\left|\int_{t^n}^{t^{n+1}}\int_{t^n}^{\sigma}\left(\frac{\partial^2 s}{\partial t^2},e^{n+1}_h-e^n_h\right)\mathrm{d}s\,\mathrm{d}\sigma\right|\\
&\qquad+\left|\int_{t^n}^{t^{n+1}}\big(\partial_t\tilde{s}_j(\sigma)-\partial_t s_j(\sigma),e^{n+1}_h-e^n_h\big)\mathrm{d}\sigma\right|\\
&\qquad+\frac{\Delta t}{\phi}\sum_j|e^{n+1}_j-e^n_j|\sum_l|G_{jl}|\\
&\quad\leqslant\frac{\Delta t}{\phi}\left[\epsilon\Upsilon\left(\sum_a[e^n_h]^2_a\right)^{1/2}+C_{31}h\|\mathbf{f}\|_{L^\infty(\Omega)}\right]\left(\sum_a[e^{n+1}_h-e^n_h]^2_a\right)^{1/2}\\
&\qquad+\Delta t\left(\int_{t^n}^{t^{n+1}}\left\|\frac{\partial^2 s}{\partial t^2}(s)\right\|_{L^2(\Omega)}\mathrm{d}s+\|\partial_t\tilde{s}-\partial_t s\|_{L^\infty(J;L^2(\Omega))}\right)\\
&\qquad\times\|e^{n+1}_h-e^n_h\|_{l^2(\Omega)}+\frac{\Delta t}{\phi}\left(\sum_a[e^{n+1}_h-e^n_h]^2_a\right)^{1/2}\Big(C_{33}h\|\mathbf{f}\|_{L^\infty(\Omega)}\Big).
\end{aligned}
$$

Dividing by $\|e^{n+1}_h-e^n_h\|_{l^2(\Omega)}$ we get:

$$
\begin{aligned}
\|e^{n+1}_h-e^n_h\|_{l^2(\Omega)}\leqslant&\frac{\sqrt{2}\Delta t}{h\phi}\left[\epsilon\Upsilon\left(\sum_a[e^n_h]^2_a\right)^{1/2}+C_{31}h\|\mathbf{f}\|_{L^\infty(\Omega)}\right]\\
&+\Delta t\left(\int_{t^n}^{t^{n+1}}\left\|\frac{\partial^2 s}{\partial t^2}(s)\right\|_{L^2(\Omega)}\mathrm{d}s+\|\partial_t\tilde{s}-\partial_t s\|_{L^\infty(J;L^2(\Omega))}\right)\\
&+C_{33}\sqrt{2}\frac{\Delta t}{\phi}\|\mathbf{f}\|_{L^\infty(\Omega)}.
\end{aligned}
$$

Finally, with Theorem 16 and triangle inequality,

$$
\begin{aligned}
&\|\partial_t\tilde{s}-\partial_t s\|^2_{L^\infty(J;L^2(\Omega))}\\
&\quad\leqslant 2\Big(\|\partial_t\tilde{s}-\partial_t I_h(s)\|^2_{L^\infty(J;L^2(\Omega))}+\|\partial_t I_h(s)-\partial_t s\|^2_{L^\infty(J;L^2(\Omega))}\Big)\\
&\quad=2\Big(\|\partial_t\tilde{s}-\partial_t I_h(s)\|^2_{L^\infty(J;l^2(\Omega))}+\|\partial_t I_h(s)-\partial_t s\|^2_{L^\infty(J;L^2(\Omega))}\Big)\\
&\quad\leqslant C_{34}h^2+C_{35}h^2\|\partial_t s\|^2_{L^\infty(J;H^2(\Omega))}.\qquad\square
\end{aligned}
$$

**Theorem 21.** *Let* $\mathbf{u}$, $\tilde{s}$ *and* $\widetilde{S}$ *be defined as in (23), (41), and (45), respectively. If* $\Delta t$ *satisfy the CFL condition*

$$
\frac{\Delta t}{h^2}<\frac{\phi}{4}\frac{\kappa}{\epsilon\Upsilon^2},\tag{68}
$$

*then we have for* $e^n_h:=\widetilde{S}^n-\tilde{s}(t^n),\ 0\leqslant n,\ N\leqslant M$, *there exist constants* $K_{13},K_{14},K_{15}>0$, *independent of* $h$ *and* $\epsilon$, *such that*:

$$\frac{1}{2}\|e_h^N\|^2_{l^2(\Omega)} \leqslant K_{13}T\left(\frac{T}{M\phi^2}+\frac{h^2}{\theta^*}\right)\|\mathbf{f}\|^2_{L^\infty(\Omega)} + \frac{T}{2}\left(\frac{T}{M}+\frac{\theta}{2}\right)(\mathscr{E}^*+K_{14}h)^2,$$

*with* $\theta := 2\phi K_{15}/\kappa\epsilon$, $\theta^* := \kappa\epsilon/2\phi$, $M\Delta t = T$ *and* $\mathscr{E}^* := \int_{t^n}^{t^{n+1}} \|\frac{\partial^2 s}{\partial t^2}(s)\|_{L^2(\Omega)}\,ds$.

**Proof.** Following the ideas of Lemma 20 and by subtraction of Eq. (41) from Eq. (45), multiplying with $e_j^n|T_j|$ and summing up over $j$ yields with $a^2-b^2-(a-b)^2 = 2(ab-b^2)$, we have

$$\frac{1}{2}(\|e_h^{n+1}\|^2_{l^2(\Omega)} - \|e_h^n\|^2_{l^2(\Omega)}) + \frac{\Delta t}{\phi}(L_h(\mathbf{u}(t^n))e_h^n, e_h^n)_h \leqslant \frac{1}{2}\|e_h^{n+1}-e_h^n\|^2_{l^2(\Omega)} + \Delta t\mathscr{E}\|e_h^n\|_{l^2(\Omega)} - \frac{\Delta t}{\phi}\sum_{a\in\mathscr{A}}[e_h]_a G_a,$$

where $G_a$ is given by Eq. (66) and $\mathscr{E} := \int_{\Delta t}\|\frac{\partial^2 s}{\partial t^2}(s)\|_{L^2(\Omega)}\,ds + \|\partial_t\tilde{s}_j - \partial_t s\|_{L^\infty(J;L^2(\Omega))}$. Applying the coerciveness of $L_h(\mathbf{u}(t^n))$, we obtain

$$\frac{1}{2}\left(\|e_h^{n+1}\|^2_{l^2(\Omega)} - \|e_h^n\|^2_{l^2(\Omega)}\right) + \frac{\Delta t}{\phi}\kappa\epsilon\sum_a [e_h^n]_a^2 \leqslant t_{10}+t_{11}+t_{12},$$

where

$$t_{10} := \frac{1}{2}\|e_h^{n+1}-e_h^n\|^2_{l^2(\Omega)},$$
$$t_{11} := \Delta t\mathscr{E}\|e_h^n\|_{l^2(\Omega)},$$
$$t_{12} := -\Delta t\sum_{a\in\mathscr{A}}[e_h]_a G_a^*,$$

with $G_a^* := g_a(\mathbf{u}(t^n); \widetilde{S}_j^n, \widetilde{S}_l^n) - g_a(\mathbf{u}(t^n); \tilde{s}_j^n, \tilde{s}_l^n)$. We have from Lemma 20 that for any $\theta, \theta^* > 0$

$$t_{10} \leqslant 2\left(\frac{\Delta t}{h}\right)^2\left(\frac{\epsilon\Upsilon}{\phi}\right)^2\sum_a [e_h^n]_a^2 + K_{13}\left(\frac{\Delta t}{\phi}\right)^2\|\mathbf{f}\|^2_{L^\infty(\Omega)} + \frac{(\Delta t)^2}{2}(\mathscr{E}^*+K_{14}h)^2,$$
$$t_{11} \leqslant \frac{K_{15}}{2}\frac{\Delta t}{\theta}\sum_a [e_h^n]_a^2 + \frac{\Delta t}{2}\theta(\mathscr{E}^*+K_{14}h)^2$$
$$t_{12} \leqslant \frac{\Delta t\theta^*}{2}\sum_a [e_h^n]_a^2 + C_{36}\frac{h^2}{2}\frac{\Delta t}{\theta^*}\|\mathbf{f}\|^2_{L^\infty(\Omega)}.$$

Taking into account that the CFL condition (68), we obtain

$$\frac{1}{2}\left(\|e_h^{n+1}\|^2_{l^2(\Omega)} - \|e_h^n\|^2_{l^2(\Omega)}\right) \leqslant K_{13}\Delta t\left(\frac{\Delta t}{\phi^2}+\frac{h^2}{\theta^*}\right)\|\mathbf{f}\|^2_{L^\infty(\Omega)} + \frac{\Delta t}{2}\left(\Delta t+\frac{\theta}{2}\right)(\mathscr{E}^*+K_{14}h)^2,$$

with $\theta := 2\phi K_{15}/\kappa\epsilon$ and $\theta^* := \kappa\epsilon/2\phi$. Finally, summing up over $n$ from 0 to $N-1$ we get the statement of the theorem. □

**Theorem 22.** *Let* $(\mathbf{U}_h, P_h, S_h)$ *be the solution of* (42)–(44), $(\tilde{\mathbf{u}}, \tilde{p}, \widetilde{S})$ *the solution of* (39)–(45) *and* $(\mathbf{u}, p, c)$ *the weak solution of* (23)–(25). *Then, there exists a constant* $K_{16} > 0$, *independent of* $h$ *and* $\epsilon$, *such that*

$$\|\mathbf{U}_h^n - \tilde{\mathbf{u}}_h^n\|_{H(\mathrm{div};\Omega)} + \|P_h^n - \tilde{p}^n\|_{L^2(\Omega)} \leqslant K_{16}\left(\|\tilde{\mathbf{u}}^n\|_{L^\infty(\Omega)}+1\right)\|s(t^n)-S_h^n\|_{L^2(\Omega)}.$$

**Proof.** The proof is the same as the proof of Theorem 18 if $(\mathbf{u}_h, p_h, s_h)$ is replaced by $(\mathbf{U}_h, P_h, S_h)$. □

**Theorem 23.** *Let* $S_h$ *be the solution of Eq.* (44) *and* $\widetilde{S}$ *the solution of Eq.* (45). *If* $\Delta t$ *satisfy the CFL condition*

$$\frac{\Delta t}{h^2} < \frac{\phi}{2}\frac{\kappa}{\epsilon\Upsilon^2} \tag{69}$$

*then for the error* $e_h^n := \widetilde{S}^n - S_h^n, 0 \leqslant n, N \leqslant M$, *there exists a constant* $K_{17} > 0$, *independent of* $h$ *and* $\epsilon$, *such that*

$$\frac{1}{2}\|e_h^N\|^2_{l^2(\Omega)} \leqslant K_{17}\frac{T}{\phi}\left(\frac{T}{M\phi}+\frac{h^2}{\epsilon\kappa}\right)\left[\|\mathbf{f}\|^2_{L^\infty(\Omega)} + \|Q\|^2_{L^\infty(\Omega)}\right]$$

*with* $M\Delta t = T$.

**Proof.** Subtracting Eq. (44) from Eq. (45), we obtain

$$e_j^{n+1} - e_j^n + \frac{\Delta t}{\phi}(L_h(\mathbf{U}_h^n)e_h^n)_j = \frac{\Delta t}{\phi}\frac{1}{|T_j|}\sum_l G_{jl} + \frac{\Delta t}{\phi}\left[(Q(s))_j - (Q(S_h^n))_j\right],$$

where $G_{jl} := g_{jl}(\mathbf{U}_h^n; S_j^n, S_l^n) - g_{jl}(\mathbf{u}(t^n); \widetilde{S}_j^n, \widetilde{S}_l^n) + g_{jl}(\mathbf{U}_h^n; e_j^n, e_l^n)$. Then, using the identity $a^2-b^2-(a-b)^2 = 2(ab-b^2)$, multiplying by $e_j^n|T_j|$ and summing over $j$, we deduce

$$\frac{1}{2}\{\|e_h^{n+1}\|^2_{l^2(\Omega)} - \|e_h^n\|^2_{l^2(\Omega)}\} + \frac{\Delta t}{\phi}(L_h(\mathbf{U}_h^n)e_h^n, e_h^n)_h = \frac{1}{2}\|e_h^{n+1}-e_h^n\|^2_{l^2(\Omega)} + \frac{\Delta t}{\phi}\sum_a [e_h^n]_a G_a + \frac{\Delta t}{\phi}\sum_j e_j^n|T_j|\left[(Q(s))_j - (Q(S_h^n))_j\right].$$

Applying the coerciveness of $L_h$ (cf. Lemma 11) , we obtain

$$\frac{1}{2}\left\{\|e_h^{n+1}\|^2_{l^2(\Omega)} - \|e_h^n\|^2_{l^2(\Omega)}\right\} + \frac{\Delta t}{\phi}\epsilon\kappa\sum_a [e_h^n]_a^2 \leqslant \frac{1}{2}\|e_h^{n+1}-e_h^n\|^2_{l^2(\Omega)} + \frac{\Delta t}{\phi}\left(\sum_a [e_h^n]_a^2\right)^{1/2}\left(\sum_a (G_a^*)^2\right)^{1/2} + \frac{\Delta t}{\phi}\sum_j e_j^n|T_j|\left[(Q(s))_j - (Q(S_h^n))_j\right],$$

where $G_a^* := g_a(\mathbf{U}_h^n; S_j^n, S_l^n) - g_a(\mathbf{u}(t^n); \widetilde{S}_j^n, \widetilde{S}_l^n)$. On the other hand, using the Eq. (67), and applying similar estimates of the proof of Lemma 20, there exists a constant $C_{37} > 0$ such that

$$\frac{1}{2}\|e_h^{n+1}-e_h^n\|_{l^2(\Omega)}^2 \leqslant \frac{\Delta t^2\epsilon^2\Upsilon^2}{\phi^2h^2}\sum_a [e_h^n]_a^2 + C_{37}\left(\frac{\Delta t}{\phi}\right)^2\left[\|\mathbf{f}\|_{L^\infty(\Omega)}^2+\|Q\|_{L^\infty(\Omega)}^2\right].$$

Therefore, using the CFL condition (69), there exists a constant $K_{17}>0$ such that

$$\frac{1}{2}\left\{\|e_h^{n+1}\|_{l^2(\Omega)}^2-\|e_h^n\|_{l^2(\Omega)}^2\right\} \leqslant K_{17}\frac{\Delta t}{\phi}\left(\frac{\Delta t}{\phi}+\frac{h^2}{\epsilon\kappa}\right)\left[\|\mathbf{f}\|_{L^\infty(\Omega)}^2+\|Q\|_{L^\infty(\Omega)}^2\right].$$

Finally, summing up over $n$ from 0 to $N-1$ we get the statement of the theorem. □

**Proof of the Theorem 10.** The proof of this result is similar to that of Theorem 9. The triangle inequality, Theorems 14 and 22 yield for all $t^n \in J_h$

Table 1
Numerical values for physical parameters

| | Symbol | Value | Unit |
|---|---|---|---|
| Absolute permeability | $k$ | $1.78\times 10^{-11}$ | [m$^2$] |
| Liquid density | $\rho_w$ | 1011 | [kg/m$^3$] |
| Gas density | $\rho_n$ | 1.16 | [kg/m$^3$] |
| Porosity | $\phi$ | 0.33 | [–] |
| Liquid viscosity | $\mu_w$ | $10^{-3}$ | [kg/m s] |
| Gas viscosity | $\mu_n$ | $1.85\times 10^{-5}$ | [kg/m s] |
| Residual water saturation | $s_{wr}$ | 0 | [–] |
| Initial water saturation | $s_w^o$ | 0.4343 | [–] |
| VG-parameter | $n$ | 1.411 | [–] |
| VG-parameter | $\alpha$ | $1.35\times 10^{-4}$ | [1/Pa] |
| Heap slope | $\theta$ | $\pi/4$ | rad. |
| Heap width | $W$ | 25 | [m] |
| Heap height | $H$ | 5 | [m] |

$$\begin{aligned}&\|\mathbf{u}(t^n)-\mathbf{U}_h^n\|_{H(\mathrm{div};\Omega)}+\|p(t^n)-P_h^n\|_{L^2(\Omega)}\\ &\quad\leqslant \|\mathbf{u}(t^n)-\tilde{\mathbf{u}}^n\|_{H(\mathrm{div};\Omega)}+\|\tilde{\mathbf{u}}^n-\mathbf{U}_h^n\|_{H(\mathrm{div};\Omega)}\\ &\qquad+\|p(t^n)-\tilde{p}^n\|_{L^2(\Omega)}+\|\tilde{p}^n-P_h^n\|_{L^2(\Omega)}\\ &\quad\leqslant \widetilde{K}_1h+\widetilde{K}_2\|s(t^n)-S_h^n\|_{L^2(\Omega)}.\end{aligned}$$

We get by the triangle inequality

$$\|s(t^n)-S_h^n\|_{L^2(\Omega)} \leqslant \|s(t^n)-I_h(s)(t^n)\|_{L^2(\Omega)}+\|I_h(s)(t^n)-\widetilde{S}^n\|_{L^2(\Omega)}+\|\widetilde{S}^n-S_h^n\|_{L^2(\Omega)}.$$

The three terms of the right hand side of this inequality can be estimate by Lemma 11, Theorems 15, 21, and 23, as follow: there exist constants $K_3,K_4,K_5>0$, such that,

$$\|s(t^n)-I_h(s)(t^n)\|_{L^2(\Omega)}^2 \leqslant K_3h^2$$

$$\begin{aligned}&\|I_h(s)(t^n)-\widetilde{S}^n\|_{L^2(\Omega)}^2\\ &\quad\leqslant 2\left(\|I_h(s)(t^n)-\tilde{s}(t^n)\|_{L^2(\Omega)}^2+\|\tilde{s}(t^n)-\widetilde{S}^n\|_{L^2(\Omega)}^2\right)\\ &\quad\leqslant 2\left(\sum_a [I_h(s)(t^n)-\tilde{s}(t^n)]_a^2+\|\tilde{s}(t^n)-\widetilde{S}^n\|_{l^2(\Omega)}^2\right)\\ &\quad\leqslant K_4\left\{\left[\frac{h^2}{(\epsilon\kappa)^2}+T\left(\frac{T}{M\phi^2}+\frac{h^2\phi}{\epsilon\kappa}\right)\right]\|\mathbf{f}\|_{L^\infty(\Omega)}^2\right.\\ &\qquad+\left(\frac{h}{\epsilon\kappa}\right)^2\|Q(s)-\phi\partial_ts\|_{L^\infty(\Omega)}^2+\frac{h^2\Upsilon^2}{\epsilon\kappa^2}\mathscr{D}(s)\\ &\qquad\left.+\frac{T}{2}\left(\frac{T}{M}+\frac{\phi}{\kappa\epsilon}\right)(\mathscr{E}^*+K_5h)^2\right\}.\end{aligned}$$

$$\|\widetilde{S}^n-S_h^n\|_{L^2(\Omega)}^2 \leqslant K_4\frac{T}{\phi}\left(\frac{T}{M\phi}+\frac{h^2}{\epsilon\kappa}\right)\left[\|\mathbf{f}\|_{L^\infty(\Omega)}^2+\|Q\|_{L^\infty(\Omega)}^2\right].$$

This completes the proof. □

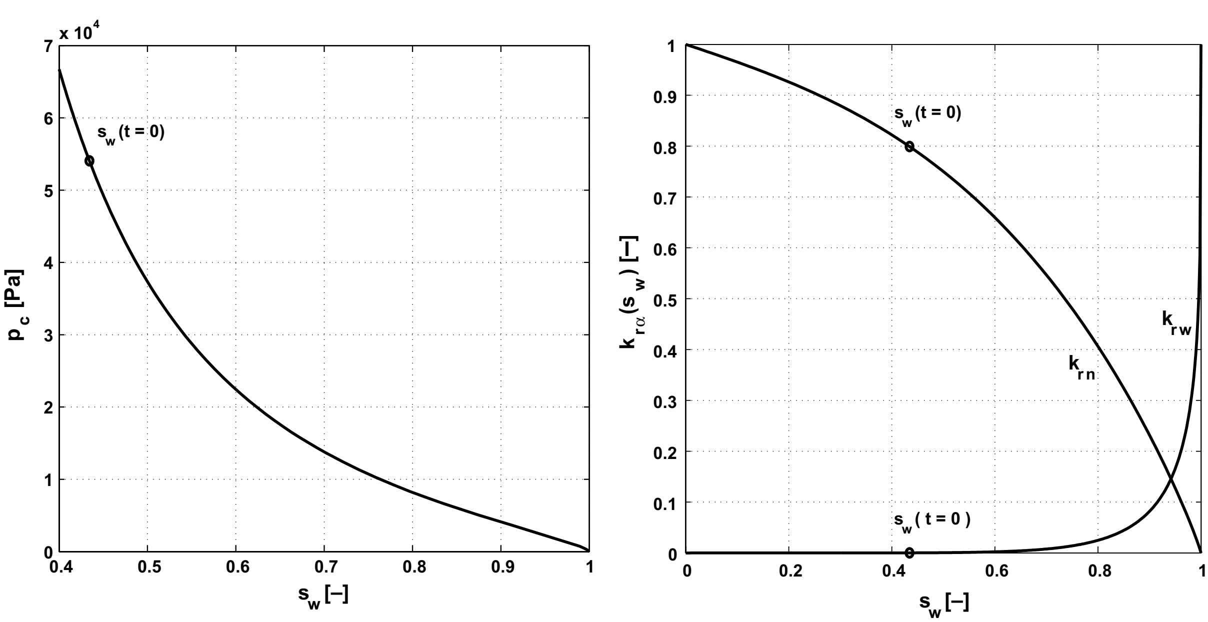


Fig. 2. Capillary pressure $p_c$ and relative permeability $k_{r\alpha}$ for VG model.

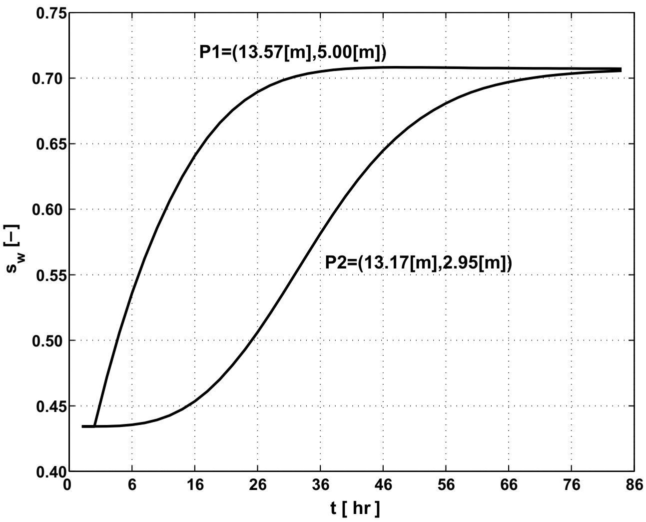


Fig. 3. Evolution of $s_w$ in two points of $\Omega$.

## 8. Numerical results

We show the behavior of our numerical scheme for the same numerical examples considered by Cariaga et al. [21,23]. The numerical solution of system (1)–(6) require an explicit definition for $p_c(\cdot)$ and $k_{r\alpha}(\cdot), \alpha = w, n$. In our simulations we prefer the Van Genuchten (VG) model (see [13]), where

$$p_c(s_w) = \frac{1}{\alpha}(S_e^{-1/m} - 1)^{1/n},$$
$$k_{rw}(s_w) = S_e^{\varepsilon}(1 - (1 - S_e^{1/m})^m)^2,$$
$$k_{rn}(s_w) = (1 - S_e)^{\gamma}(1 - S_e^{1/m})^{2m},$$

with $S_e(s_w) := \frac{s_w - s_{wr}}{1 - s_{wr}}$ the effective saturation and $s_{wr}$ the residual water saturation [13]. The terms $\varepsilon$ and $\gamma$ are form parameters which describe the connectivity of the pores. Generally, $\varepsilon = \frac{1}{2}$ and $\gamma = \frac{1}{3}$. For an analysis of (VG) parameters, in the heap leaching context (see [21]). In Table 1 we show our choice of parameters in the heap leaching context. Our choice is similar to that of Li [22]. On the other hand, our computational code consider an implicit scheme for the MFE method to obtain an approximation of $p(\mathbf{x}, t^{n+1})$ and $\mathbf{u}(\mathbf{x}, t^{n+1})$, where the liquid saturation $s_w$ is replaced by an approximation of $s_w(\mathbf{x}, t^n)$, while that the saturation equation is solved by a cell centered FV implicit scheme to obtain an approximation of $s_w(\mathbf{x}, t^{n+1})$, where the total velocity $\mathbf{u}$ is replaced by an approximation of $\mathbf{u}(\mathbf{x}, t^{n+1})$, [23,24]. We use a damped inexact Newton algorithm for solving the non-linear system of equations, [25,13]. The capillary pressure, $p_c$ and the relative permeability, $k_{r\alpha}$ used are plotted in Fig. 2. A plot of the evolution of $s_w$ in two points $P1 = (13.57, 5.00)$ and $P2 = (13.17, 2.95)$ in the heap $\Omega$ is given in Fig. 3, for an irrigation ratio $R = 5.34[lt/hr/m^2]$.

## Acknowledgements

The authors wish to thank Fondecyt projects No. 1040597 and No. 1030718, Conicyt-DAAD project No. 2003-154, Fondap in Applied Mathematics No. 15000001, and the Universidad Católica de Temuco, for the financial support given to this research work.

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